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A Global Optimization Method for Solving Convex Quadratic Bilevel Programming Problems

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Abstract. We use the merit function technique to formulate a linearly constrained bilevel convex quadratic problem as a convex program with an additional convex-d.c. constraint. To solve the latter problem we approximate it by convex programs with an additional convex-concave constraint using an adaptive simplicial subdivision. This approximation leads to a branch-and-bound algorithm for finding a global optimal solution to the bilevel convex quadratic problem. We illustrate our approach with an optimization problem over the equilibrium points of an n-person parametric noncooperative game.

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1. Introduction

Bilevel programming involves two optimization problems where the constraint region of the first-level problem is implicitly determined by another second-level (inner) optimization problem. Let $f_1, f_2 : \mathbb{R}^{m+n} \to \mathbb{R}$, $T \subset \mathbb{R}^m$, $X \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^{m+n}$. Then the bilevel problem can be formulated as

$$\min\{f_1(t, x) : t \in T\} \tag{1.1}$$

where x solves

$$\min\{f_2(t, x) : x \in X, (t, x) \in C\}. \tag{1.2}$$

Due to its nested structure a bilevel programming problem, even in the linear case, i.e, both the first and second-level problems are linear, is a nonconvex optimization problem. Bilevel programming has received increasing attention in the literature and some algorithms have been developed (see, e.g., [1,4–7,14,17,20] and the references therein). Most algorithms in this field have been obtained for the bilevel linear problem. In this case it is proved that a solution to the problem must occur at an extreme of the polyhedron C. This result is extended in [9] for the case where both f_1 and f_2 are quasiconcave. Based on this fact, several algorithms have been proposed for finding an optimal solution by enumerative schemes [3,17]. A survey of the linear bilevel programming is provided in [8]. Recently, global optimization methods such as branch-and-bound, cutting plane and outer approximation have been proposed for solving bilevel linear problems [14,22,24,26,27,29]. An approach used in bilevel linear problems is to use Kuhn-Tucker optimality conditions to rewrite the bilevel problem as a single level problem. The same approach may be used for converting more general convex bilevel problems, including convex quadratic bilevel problems. Branch-and-bound methods [5,7] and descent algorithms [25,28] have been proposed for solving convex bilevel problems based on this reformulation. The latter methods are rather confined to computing stationary points or local minima.

In this paper we consider the problem (1.1) and (1.2) where T, X, C are polyhedral convex sets and $f_1(t,x)$, $f_2(t,x)$ are convex quadratic functions. Unlike the linear case, this problem does not necessarily attain its optimal solution at an extreme point of C. We use the merit function technique [1,12] to formulate the problem as a convex program with an additional nonconvex constraint defined by a function which is d.c with respect to the decision variable of the first level-problem and convex with respect to the decision variable of the second level-problem. To solve the latter problem we approximate it by convex programs with an additional convex–concave constraint and solve the resulting problems by a branch-and-bound procedure using an adaptive simplicial subdivision performed in the t-space. The algorithm therefore is designed for Problem (P) where the dimension of t is relatively small. The dimension of x may be larger. We illustrate our approach with an optimization problem over the equilibrium points of an oligopolistic market problem which is equivalent to an n-person noncooperative parametric game.

2. Equivalence Formulation

In what follows we consider Problem (1.1)–(1.2) where f_1, f_2 are convex quadratic functions, and $T \subset \mathbb{R}^m$, $X \subset \mathbb{R}^n$, $C \subset \mathbb{R}^{m+n}$ are polyhedral convex sets. Suppose

$$C := \{(t,x) : At + Bx + b \leqslant 0\},$$

$$f_2(t,x) := \frac{1}{2}x^T Qx + x^T(Pt + q),$$

where $b \in \mathbb{R}^\ell$, $q \in \mathbb{R}^n$, $A \in \mathbb{R}^{\ell\times m}$, $B \in \mathbb{R}^{\ell\times n}$, $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{n\times n}$. We suppose further that the polyhedron X is bounded and that Q is positive definite. Without loss of generality we may assume that Q is symmetric, since we may replace, if necessary, Q by $1/2(Q + Q^T)$.

For each fixed t we define

$$C(t) := \{x \in X : At + Bx + b \leqslant 0\}.$$

The bilevel convex problem (P) under consideration then can be formulated as

$$\min\{f_1(t,x) : t \in T\} \tag{P}$$

where x solves

$$\min\left\{f_2(t,x) := \frac{1}{2}x^T Qx + x^T(Pt+q) : x \in C(t)\right\}. \tag{$P(t)$}$$

We recall that a point (t, x) is said to be feasible for Problem (P) if $t \in T$ and x is an optimal solution to the second level-problem (P(t)).

We define a merit function for this problem by setting

$$g(t,x) := \frac{1}{2}x^T Qx + x^T(Pt+q) - \min_v \left\{\frac{1}{2}v^T Qv + v^T(Pt+q) : v \in C(t)\right\}. \tag{2.1}$$

The following lemma is immediate from the definition of g.

LEMMA 2.1. *Suppose that for each fixed t the second-level problem (P(t)) admits a solution. Then*

(i) $g(t,x) \geqslant 0 \;\; \forall t \in R^m, \;\; \forall x \in C(t)$.

(ii) $g(t,x) = 0, t \in T, \;\; x \in C(t)$ *if and only if x is an optimal solution to (P(t)).*

From Lemma 2.1 it follows that Problem (P) can be formulated by the following one-level problem

$$\min\{f_1(t,x) : t \in T, x \in X, g(t,x) \leqslant 0\}. \tag{$Q1$}$$

This is a nonconvex problem, since the function $g(t, x)$ is not convex. In order to solve this problem we need to further analyse the function g. To this end, for each t we take

$$\varphi(t) := \min_v \left\{\frac{1}{2}v^T Qv + v^T(Pt+q) : v \in C(t)\right\}. \tag{2.2}$$

Then

$$g(t,x) = \frac{1}{2}x^T Qx + x^T(Pt+q) - \varphi(t). \tag{2.3}$$

The following simple one-dimensional example shows that φ is neither convex nor concave.

Let φ be given by

$$\varphi(t) := \min_v \{v^2 + (t+1)v : \; v \geqslant t+1\}.$$

A simple calculation shows that

$$\varphi(t) = \begin{cases} 2(t+1)^2 & \text{if } \; t \geqslant -1, \\ -\dfrac{1}{4}(t+1)^2 & \text{if } \; t \leqslant -1. \end{cases}$$

Clearly $\varphi(t)$ is neither convex nor concave on R. So the function $g(t,x)$ given by (2.3), in general, is neither convex nor concave.

The next lemma gives an equivalent formulation for the merit function g that is useful for deriving solution methods for Problem (P).

LEMMA 2.2. *Suppose that Q is symmetric positive definite and $C(t)$ is nonempty and bounded for each t. Then*

$$\begin{aligned} g(t,x) \;=\; & \frac{1}{2}x^T Qx + x^T(Pt+q) + \frac{1}{2}t^T(P^TQ^{-1}P)t \\ & + t^T(P^TQ^{-1}q) + \frac{1}{2}q^TQ^{-1}q + r(t) \end{aligned} \tag{2.4}$$

where $r(t)$ is a finite concave function given by

$$r(t) := \min_{\lambda \geqslant 0}\left\{\frac{1}{2}\lambda^T(BQ^{-1}B^T)\lambda - \lambda^T((A-BQ^{-1}P)t + b - BQ^{-1}q)\right\}. \tag{2.5}$$

Proof. Applying Kuhn-Tucker theorem to Problem (2.2) defining $\varphi(t)$ we have

$$\begin{aligned} \varphi(t) \;&= \max_{\lambda \geqslant 0}\left\{\min_{v\in R^n}\left\{\frac{1}{2}v^TQv + v^T(Pt+q)\right\} + \lambda^T(At+Bv+b)\right\} \\ &= \max_{\lambda \geqslant 0}\left\{\lambda^T(At+b) + \min_{v\in R^n}\left\{\frac{1}{2}v^TQv + v^T(Pt+B^T\lambda+q)\right\}\right\}. \end{aligned}$$

Let

$$\psi(t) := \min_{v\in R^n}\left\{\frac{1}{2}v^TQv + v^T(Pt+B^T\lambda+q)\right\}.$$

Since Q is symmetric positive definite, the latter problem has a unique solution v^* given by

$$v^* = -Q^{-1}(Pt + B^T\lambda + q).$$

A simple calculation shows that

$$\begin{aligned} \psi(t) \;=\; & -\frac{1}{2}(t^T(P^TQ^{-1}P)t + \lambda^T(BQ^{-1}B^T)\lambda + q^TQ^{-1}q) \\ & -\lambda^T(BQ^{-1})(Pt+q) - t^T(P^TQ^{-1}q). \end{aligned}$$

Thus

$$\varphi(t) = -\frac{1}{2}t^T(P^TQ^{-1}P)t - \frac{1}{2}q^TQ^{-1}q - t^T(P^TQ^{-1}q) - r(t).$$

where

$$r(t) = \min_{\lambda \geqslant 0} \left\{ \frac{1}{2}\lambda^T (BQ^{-1}B^T)\lambda - \lambda^T((A - BQ^{-1}P)t + b - BQ^{-1}q) \right\}. \tag{2.6}$$

Thus

$$\begin{aligned} g(t, x) \ = \ & \frac{1}{2}x^T Qx + x^T(Pt + q) + \frac{1}{2}t^T(P^T Q^{-1}P)t + t^T(P^T Q^{-1}q) \\ & + \frac{1}{2}q^T Q^{-1}q + r(t). \end{aligned}$$

Since $\varphi(t)$ is finite, $r(t)$ is finite too. □

By Lemma 2.2 and (Q1), Problem (P) can be reformulated as

$$f_1(P) := \ \min f_1(t, x) \tag{$P1$}$$

subject to

$$t \in T, x \in X, \tag{2.7}$$

$$At + Bx + b \leqslant 0, \tag{2.8}$$

$$\frac{1}{2}x^T Qx + x^T(Pt + q) + \frac{1}{2}t^T(P^T Q^{-1}P)t + t^T(P^T Q^{-1}q) + \frac{1}{2}q^T Q^{-1}q + r(t) \leqslant 0, \tag{2.9}$$

where $r(t)$ is the concave function given by (2.6).

Clearly, by Lemmas 2.1 and 2.2, a point (t, x) is feasible for (P1) if and only if it is feasible for (P).

3. Solution Method

It is well recognized that the branch-and-bound technique has been applied successfully for solving a lot number of global optimization problems. Branch-and-bound methods differ from each other by the rules they use for bounding and branching. In this section we propose a branch-and-bound algorithm for solving (P1) which is a nonconvex programming problem because of constraint (2.9). We must first define bounding and branching operations in order to develop a branch-and-bound algorithm.

3.1. BOUNDING AND BRANCHING

We use an approximation of the nonconvex constraint (2.9) in order to compute lower bounds for Problem (P1). Specially, we approximate the constraint (2.9) by a convex-concave constraint using a linear approximation of the convex quadratic function $1/2t^T(P^TQ^{-1}P)t$ over a simplex in the t-space. Then we use a suitable simplicial subdivision to refine the approximation. More precisely, let us consider Problem (P1) restricted on a simplex S, i.e.,

$$f_1(S) := \min f_1(t, x) \tag{PS}$$

subject to

$$\begin{aligned}&t \in S \cap T, x \in X,\\ &At + Bx + b \leqslant 0,\\ &\frac{1}{2}x^TQx + x^T(Pt + q) + \frac{1}{2}t^T(P^TQ^{-1}P)t\\ &+t^T(P^TQ^{-1}q) + \frac{1}{2}q^TQ^{-1}q + r(t) \leqslant 0.\end{aligned}$$

Let $u \in S$ be fixed. Define a linearization $l_u(t)$ of the function $\theta(t) := 1/2t^T(P^T Q^{-1}P)t$ at u by setting

$$l_u(t) := u^T(P^TQ^{-1}P)t - \frac{1}{2}u^T(P^TQ^{-1}P)u. \tag{3.1}$$

Clearly $l_u(t) \leqslant \theta(t)$ for all t and $l_u(u) = \theta(u)$. Then take

$$g_u(t, x) := \frac{1}{2}x^TQx + x^T(Pt + q) + l_u(t) + t^T(P^TQ^{-1}q) + \frac{1}{2}q^TQ^{-1}q + r(t). \tag{3.2}$$

Since Q is positive definite and $r(t)$ is concave, this function is convex in x for each fixed t and concave in t for each fixed x. Moreover from $l_u(t) \leqslant \theta(t)$ it follows that

$$g(t, x) \leqslant 0 \Rightarrow g_u(t, x) \leqslant 0.$$

The following lemma can be derived from Lemma 2.3 in [21] (see also [16]).

LEMMA 3.1. *Denote by $V(S)$ the set of the vertices of S. Let*

$$\begin{aligned}\beta(v) \ := \ &\min\{f_1(t, x) : t \in S \cap T, x \in X,\\ &At + Bx + b \leqslant 0, g_u(v, x) \leqslant 0\}\end{aligned} \tag{$P(v)$}$$

and

$$\beta(S) := \min\{\beta(v) : v \in V(S)\}.$$

Then $\beta(S) \leqslant f_1(S)$.

We will apply this lemma with $u = u^S \in V(S)$. We shall refer to u^S as *approximation point* for S. Let v^S be the vertex of S corresponding to $\beta(S)$ and let (t^S, x^S) be an optimal solution to Problem (P(v^S)). Then

$$\beta(S) = \beta(v^S) = f_1(t^S, x^S),\ \ g_{u^S}(v^S, x^S) \leqslant 0.$$

Clearly, if $t^S = v^S = u^S$, then (t^S, x^S) is feasible for (PS) because

$$x^S \in X, At^S + Bx^S + b \leqslant 0, g(t^S, x^S) = g_{u^S}(t^S, x^S) \leqslant 0.$$

In this case $\beta(S) = f_1(t^S, x^S) = f_1(S)$ which means that a solution to Problem (PS) has been found. The simplex S then can be eliminated from further consideration. This suggests the use of a simplicial subdivision such that the iteration points t^S, u^S and v^S tend to the same point as the algorithm runs infinitely many times. To define this subdivision, suppose that among three points t^S , u^S and v^S there are at least two distinct points. We then subdivide S as follows.

Let ω^S be the midpoint of the longest line segment among the segments $[t^S, u^S]$, $[u^S, v^S]$, $[v^S, t^S]$. Note that $\omega^S \notin V(S)$, since there are at least two distinct points among t^S , u^S and v^S. Then we subdivide the simplex S by a radial subdivision [15] that is defined as follows:

Let v^j $(j = 1, ..., m+1)$ be the vertices of S. Then ω is uniquely expressed as

$$\omega = \sum_{j=1}^{m+1} \lambda_j v^j,\ \sum_{j=1}^{m+1} \lambda_j = 1,\ \lambda_j \geqslant 0\, \forall j.$$

Let

$$J(\omega) := \{j : \lambda_j > 0\}.$$

As usual we shall refer to ω as the *subdivision point* and to $J(\omega)$ as the *subdivision indices* for S. Note that $J(\omega)$ has at least two elements since $\omega \notin V(S)$. We then subdivide S into simplices S_j, $j \in J(\omega)$, where the simplex S_j is obtained from S by replacing the vertex v^j by ω.

The main advantage of this simplicial subdivision is that it takes into account information obtained from bounding operation. However, a branch-and-bound algorithm utilizing a pure radial simplicial subdivision is not guaranteed to converge. To ensure the convergence in the algorithm to be presented below we shall combine this adaptive subdivision with the exhaustive bisection via the midpoint of a longest edge of the partition simplex [15] as follows:

SUBDIVISION RULE 1. [1, 15]. Let $N \geqslant 1$ be a natural number chosen in advance. Let S_k be the simplex to be subdivided at iteration k and let t^k, u^k, v^k be the iteration points obtained when computing the lower bound $\beta(S_k)$ according to Lemma 3.1. If k is a multiplier of N, then we take the subdivision point ω^k as the midpoint of a longest edge of S_k. Otherwise take ω^k as the midpoint of a longest line segment among the segments $[t^S, u^S]$, $[u^S, v^S]$ $[v^S, t^S]$.

Clearly if $N = 1$ then at every iteration the subdivision point is the midpoint of a longest edge of the partition simplex. In this case we have an exhaustive simplex bisection [15].

Computing upper bounds. The feasible region of Problem (P) is a nonconvex set. However unlike mathematical programming problems having nonconvex feasible domain, a feasible point of this problem can be obtained by solving a linearly constrained convex quadratic problem. In fact for fixed t^S we compute a solution of the second-level problem

$$\min_x\{f_2(t^S, x) := \frac{1}{2}x^T Qx^T + x^T(Pt^S + q) : \ x \in X, At^S + Bx + b \leqslant 0\}.$$

Since the function $f_2(t^S, .)$ is positive definite quadratic, this problem has a unique solution, say y^S. Hence (t^S, y^S) is feasible for bilevel problem (P) and therefore, by Lemma 2.1, it is feasible for (P1). Thus $f_1(t^S, y^S)$ is an upper bound for the optimal value of (P).

The algorithm may be described in detail now that the bounding and branching operations have been defined. As usual a feasible point (t^*, x^*) is said to be an ϵ-global optimal solution to Problem (P) if it is feasible for (P) and

$$f_1(t^*, x^*) - f_1(P) \leqslant \epsilon(|f_1(t^*, x^*)| + 1).$$

ALGORITHM

Start. Choose a natural number $N \geqslant 1$ and a tolerance $\epsilon \geqslant 0$.

Construct an $m + 1$- simplex S_0 containing T (methods for constructing such a simplex can be found, for example, in [15]).

Take $u^0 \in S_0$. For each $v \in V(S_0)$ solve the convex quadratic program

$$\begin{array}{ll}\beta(v) & := \min\{f_1(t, x) : t \in T, x \in X, \\ At + Bx + b \leqslant 0, \ g_u(v, x) \leqslant 0\} & \end{array} \qquad (P(v))$$

with $u = u^0$, and set

$$\beta_0 := \min\{\beta(v) : v \in V(S_0)\} = \beta(v^0).$$

Let (t^0, x^0) be the obtained solution of (P(v^0)). Compute an upper bound α_0 by setting

$$\alpha_0 := f_1(t^0, y^0),$$

where y^0 is the unique solution of convex quadratic problem

$$\min_x\{f_2(t^0, x) : x \in X, At^0 + Bx + b \leqslant 0\}.$$

Set $(\bar{t}^0, \bar{x}^0) := (t^0, y^0)$ (the currently best feasible point of (P1)) and

$$\Gamma_0 = \begin{cases} \{S_0\} & \text{if } \alpha_0 - \beta_0 > \epsilon(|\alpha_0| + 1), \\ \emptyset & \text{otherwise}. \end{cases}$$

Iteration k ($k = 0, 1, ...$)
If $\Gamma_k = \emptyset$ then terminate: $(\bar{t}^k, \bar{x}^k)$ is an ϵ-global optimal solution to (P1).
If $\Gamma_k \neq \emptyset$, then select $S_k \in \Gamma_k$ such that

$$\beta(S_k) = \min\{\beta(S) : S \in \Gamma_k\}.$$

Let $(t^{S_k}, u^{S_k}, v^{S_k})$ be the iteration points corresponding to $\beta(S_k)$ computed by Lemma 3.1 (for simplicity we shall write (t^k, u^k, v^k) for $(t^{S_k}, u^{S_k}, v^{S_k})$). Define the subdivision ω^k as the midpoint of a longest segment among the segments $[t^k, u^k]$, $[u^k, v^k]$ and $[v^k, t^k]$.

Subdivide S_k into subsimplices S_{kj} $(j \in J_k)$ according to the subdivision rule 1 (J_k denotes the subdivision indeces for S_k).

For each newly generated S_{kj} compute $\beta(S_{kj})$ according to Lemma 3.1 with the approximation point $u^{kj} = \omega^k$ for all $j \in J_k$ to obtain a triple (t^{kj}, u^{kj}, v^{kj}). Solve the convex programs

$$\min_x\{f_2(t^{kj}, x) : x \in X, At^{kj} + Bx + b \leqslant 0\} \ (j \in J_k).$$

Let $y^{kj} (j \in J_k)$ be the solutions of these problems (hence (t^{kj}, y^{kj}) is feasible for (P1)), and $\alpha(S_{kj}) := f_1(t^{kj}, y^{kj})$, $j \in J_k$.

Update the currently best upper bound by setting

$$\alpha_{k+1} := \min\{\alpha_k, \ \alpha(S_{kj}) : j \in J_k\}.$$

Let $(\bar{t}^{k+1}, \bar{x}^{k+1})$ be the point among $(\bar{t}^k, \bar{x}^k)$ and (t^{kj}, y^{kj}) with $j \in J_k$ such that $\alpha_{k+1} = f_1(\bar{t}^{k+1}, \bar{x}^{k+1})$.

Set

$$\Gamma_{k+1} := \{S \in \{(\Gamma_k \setminus S_k) \bigcup_{j \in J_k} S_{kj}\} \ : \ \alpha_{k+1} - \beta(S) > \epsilon(|\alpha_{k+1}| + 1)\}.$$

Increase k by 1 and return to iteration k.

REMARK 3.1. In the algorithm, N may be an arbitrary natural number. In practice the algorithm mainly uses adaptive simplicial subdivision. The bisection via a longest edge of the partition simplex is used only when the currently best lower bound is very slowly improved.

REMARK 3.2. Since the subdivision of the algorithm takes place in the t-space, the algorithm is designed for Problem (P) where the dimension of t is relatively small. The dimension of x may be larger.

CONVERGENCE THEOREM. *(i) If the algorithm terminates at iteration k, then* $(\bar{t}^k, \bar{x}^k)$ *is an* ϵ*-global optimal solution of (P).*

(ii) If the algorithm runs with infinitely many iterations, then $\alpha_k \searrow f_1(P)$, $\beta_k \nearrow f_1(P)$, *and any cluster point of the sequence* $\{(\bar{t}^k, \bar{x}^k)\}$ *solves (P) globally.*

Proof. (i) The algorithm terminates at iteration k if and only if $\Gamma_k = \emptyset$. This implies that $\alpha_k - \beta_k \leqslant \epsilon(|\alpha_k|+1)$. Since $(\bar{t}^k, \bar{x}^k)$ is feasible for (P), $\alpha_k = f_1(\bar{t}^k, \bar{x}^k)$ and β_k is a lower bound for $f_1(P)$, we deduce that $(\bar{t}^k, \bar{x}^k)$ is an ϵ-global optimal solution to (P).

(ii) Suppose now that the algorithm runs infinitely many iterations. By Proposition VII.5 [15] it generates an infinite nested sequence $\{S_j\}$ of simplices such that the length of the longest segment among the segments $[t^j, u^j]$, $[u^j, v^j]$ and $[v^j, t^j]$ tends to zero as $j \to \infty$. This implies that t^j, u^j and v^j tend to the same limit, say $t^* \in T$. Since $\beta(v^j)$ is the optimal value of Problem (P(v^j)), we have

$$\beta_j = \beta(S_j) = f_1(t^j, x^j), \tag{3.3}$$

$$x^j \in X, At^j + Bx^j + b \leqslant 0, \;\; g_{u^j}(v^j, x^j) \leqslant 0. \tag{3.4}$$

By the definition of $g_u(t, x)$ we have

$$\begin{aligned} g_{u^j}(t^j, x^j) = {} & \frac{1}{2}(x^j)^T Q x^j + (x^j)^T)(Pt^j + q) + (u^j)^T (P^T Q^{-1} P) t^j \\ & +(t^j)^T (P^T Q^{-1} q) + \frac{1}{2} q^T Q^{-1} q - \frac{1}{2}(u^j)^T (P^T Q^{-1} P) u^j + r(t^j) \leqslant 0. \end{aligned} \tag{3.5}$$

Since X is compact, we may assume, taking a subsequence if necessary, that $x^j \to x^* \in X$. Since $f_1(P1) \geqslant \beta_{k+1} \geqslant \beta_k$ for all k, from (3.3), (3.4) and (3.5) we obtain in the limit that

$$\beta_* = \lim_k \beta_k = f_1(t^*, x^*) \leqslant f_1(P1), \tag{3.6}$$

$$x^* \in X, At^* + Bx^* + b \leqslant 0, \tag{3.7}$$

$$\begin{aligned} & \frac{1}{2}(x^*)^T Q x^* + (x^*)^T)(Pt^* + q) + (u^*)^T (P^T Q^{-1} P) t^* \\ & +(t^*)^T (P^T Q^{-1} q) + \frac{1}{2} q^T Q^{-1} q - \frac{1}{2}(u^*)^T (P^T Q^{-1} P) u^* + r(t^*) \leqslant 0. \end{aligned} \tag{3.8}$$

Since $t^* = u^*$, we have

$$(u^*)^T (P^T Q^{-1} P) t^* - \frac{1}{2}(u^*)^T (P^T Q^{-1} P) u^* = \frac{1}{2}(u^*)^T (P^T Q^{-1} P) u^*.$$

Then from (2.4) and (3.8) it follows that $g(t^*, x^*) \leqslant 0$ which together with (3.7) shows that (t^*, x^*) is feasible for (P1), and therefore, by (3.6), it solves (P1). In particular, x^* solves the second-level problem (P(t^*)).

On the other hand according to the rule for computing upper bounds, y^j is the optimal solution of (P(t^j)). As before we may assume that $y^j \to y^*$ as $j \to \infty$. Since the second-level problem (P(t^j)) is convex quadratic, and $t^j \to t^*$, it follows from Theorem 5.3.2 in [2] that y^* solves (P(t^*)). Since this problem has a unique solution, we get $x^* = y^*$.

Note that

$$\beta_j = f_1(t^j, x^j) \leqslant f_1(P1)\ \forall j.$$

But since $(\bar{t}^j, \bar{x}^j)$ is the currently best feasible point at iteration j, we have

$$f_1(P1) \leqslant \alpha_j = f_1(\bar{t}^j, \bar{x}^j) \leqslant f_1(t^j, y^j)\ \ \forall j.$$

Letting $j \to \infty$ and we obtain

$$\lim_j \beta_j = f_1(t^*, x^*) \leqslant f_1(P1) \leqslant \lim_j \alpha_j \leqslant f_1(t^*, y^*)$$

which together with $y^* = x^*$ implies

$$\lim_j \beta_j = f_1(t^*, x^*) = \lim_j \alpha_j = f_1(P1).$$

Thus (t^*, x^*) is an global optimal solution to (P1). Note that the sequences $\{\beta_k\}$ and $\{\alpha_k\}$ are monotone, we have $\beta_k \nearrow f_1(P1)$, $\alpha_k \searrow f_1(P1)$.

Now let $(\bar{t}^*, \bar{x}^*)$ be any cluster point of the sequence $\{(\bar{t}^k, \bar{x}^k)\}$. Let $\{(\bar{t}^j, \bar{x}^j)\}$ be a subsequence that converges to $(\bar{t}^*, \bar{x}^*)$. Since $(\bar{t}^j, \bar{x}^j)$ is the best feasible point at iteration j and $f_1(\bar{t}^j, \bar{x}^j) = \alpha_j$, we obtain in the limit that

$$f_1(\bar{t}^*, \bar{x}^*) = \lim_j \alpha_j = f_1(P).$$

Noting that $(\bar{t}^j, \bar{x}^j)$ is feasible for every j we deduce that $(\bar{t}^*, \bar{x}^*)$ is a global optimal solution to (P1), and hence, by Lemma 2.1, it is a global optimal solution to (P). The theorem is proved. □

REMARK 3.3. The validity of the algorithm and its convergence theorem remains to hold if $f_1(t, x)$ is an arbitrary continuous convex function. In this case the subprograms needed to solve in the algorithm remain convex but not quadratic.

A Special Case. Consider the special case where the second-level problem does not depend on the decision variable t of the first-level problem. That is, $A \equiv 0$ in constraint (2.8) so that

$$C(t) \equiv C_0 = \{x \in X, Bx + b \leqslant 0\}$$

is independent of t.

By Lemma 2.1 we have

$$g(t, x = \frac{1}{2}x^T Qx + x^T(Pt + q) - \varphi(t),$$

where

$$\varphi(t) = \min_v \left\{ \frac{1}{2} v^T Q v + v^T (Pt + q) : v \in C_0) \right\}. \tag{3.9}$$

Since $\varphi(t)$ now is the minimum of a family of affine functions, it is a concave function. Thus $g(t, x)$ is a biconvex function, i.e., convex in x for every fixed t and convex in t for every fixed $x \in X$.

Problem (P) then can be reformulated as

$$\min\{f_1(t, x) : \; t \in T\}$$

subject to

$$x \in X, Bx + b \leqslant 0,$$
$$g(t, x) = \frac{1}{2} x^T Q x + x^T (Pt + q) - \varphi(t) \leqslant 0.$$

For this case in the above algorithm, instead of linearizing the convex function $1/2t^T(P^T Q^{-1} P)t$ we can use the linearization of the convex function $-\varphi(t)$. Namely instead of the function $l_u(t)$ defined by (3.1) we take

$$l_u(t) = \langle w, t - u \rangle - \varphi(u), \tag{3.1$'$}$$

where w is a gradient of the convex function $-\varphi(t)$ at u. Then the function $g_u(t, x)$ defined by (3.2) now takes the form

$$g_u(t, x) = \frac{1}{2} x^T Q x + x^T (Pt + q) + l_u(t). \tag{3.2$'$}$$

This function is convex in x for each fixed t and linear in t for each fixed x. Note that since the convex quadratic problem (3.9) defining $\varphi(t)$ is uniquely solvable, the function $\varphi(t)$ is differentiable. Moreover if v is the solution of problem (3.9) defining $\varphi(u)$, then it is easy to verify that $w = v^T P$ is the gradient of $\varphi(t)$ at u.

4. Illustrative Example

We illustrate Problem (P) and the proposed algorithm with the optimization problem over the equilibrium points of an oligopolistic market linear equilibrium model (see [18]). Suppose that there are n firms (followers) producing a goods and that the price p of the goods depends on its quantity and on control parameters $t = (t_1, ..., t_m)$ representing, for example, the export and import duties or petrol price on the world market. Let denote by x_j the quantity of the goods producing by firm j for $j = 1, ..., n$.

Following [18] we suppose that the price and the cost function are given respectively by

$$\begin{aligned} p(t, x) \; &= \; \alpha + c^T t - \beta \sum_{j=1}^{n} x_j, \\ h_j(t, x_j) \; &= \; (d_j^T t + \gamma_j) x_j + \delta_j, \;\; j = 1, ..., n, \end{aligned}$$

where $\gamma_j > 0, \beta > 0, \delta_j \geqslant 0$, and $d_j \in R^m$. Then the utility function of firm j is defined by

$$u_j(t, x_1, ..., x_n) = x_j p(t, x) - h_j(t, x). \tag{4.1}$$

Suppose that to produce the goods the firms need ℓ different materials. Denote by b_{kj} the quantity of material k that firm j needs to produce a unit of the goods, and by b_k the quantity of material k can be ordered. The constraints then are

$$t_i \in T_i := \{t_i : 0 \leqslant t_i \leqslant \tau_i < \infty\} \; i = 1, ..., m, \tag{4.2}$$

$$x_j \in X_j := \{x_j : 0 \leqslant x_j \leqslant \xi_j < \infty\} \; j = 1, ..., n, \tag{4.3}$$

$$\sum_{j=1}^{n} b_{kj} x_j \leqslant b_k \; k = 1, ..., \ell, \tag{4.4}$$

where τ_i and ξ_j stand for the upper bounds for the control parameter i and goods j respectively.

As usual we call $(x_1^*, ..., x_n^*)$ an equilibrium point for the oligopolistic market problem with respect to a control parameter t if

$$\begin{aligned} &u_j(t, x_1^*, ..., x_{j-1}^*, y_j, x_{j+1}^*, ..., x_n^*) \\ &\leqslant u_j(t, x_1^*, ..., x_{j-1}^*, x_j^*, x_{j+1}^*, ..., x_n^*) \; \forall \mu_j \leqslant y_j \leqslant \xi_j, \; j = 1, ..., n \end{aligned}$$

It follows from Proposition 3.2.6 in [18] that $(x_1^*, ..., x_n^*)$ is equilibrium with respect to t if and only if it is the optimal solution to the convex quadratic problem

$$\min_x \left\{ f_2(t, x) := \frac{1}{2} x^T Q x^T + x^T (Pt + q) \right\}$$

subject to constraints (4.3) and (4.4), where Q is the $n \times n$-symmetric positive definite matrix given by

$$Q = \begin{pmatrix} 2\beta & \beta & ... & \beta \\ \beta & 2\beta & ... & \beta \\ ... & ... & ... & ... \\ \beta & \beta & ... & 2\beta \end{pmatrix} \tag{4.5}$$

and P is the $n \times m$ matrix whose p_{ji} entry is given by

$$p_{ji} = d_{ji} - c_i, \; i = 1, ..., m, \; j = 1, ..., n \tag{4.6}$$

and

$$q = \gamma - \alpha e \in R^n. \tag{4.7}$$

Let $f_1(t, x)$ denote the objective function in the first-level problem. Then the optimization problem over the set of the equilibrium points of the above oligopolistic linear market problem can be formulated as the bilevel problem

$$\min\{f_1(t, x) : t \in T = T_1 \times T_2 ... \times T_m\}$$

where x solves the convex quadratic problem

$$\min_x\{f_2(t, x) := \frac{1}{2}x^T Qx^T + x^T(Pt + q)\}$$

subject to

$$x \in X := X_1 \times X_2 .... \times X_n,$$

$$\sum_{j=1}^{n} b_{kj}x_j \leqslant b_k \; k = 1, ..., \ell$$

with Q, P and q being given by (4.5), (4.6) and (4.7) respectively. Clearly, in this model the feasible domain of the second level-problem does not depend on the decision variable t in the first level-problem. Thus, from the results in the preceding sections, this problem can be formulated equivalently by the following one-level optimization problem

$$\min f_1(t, x)$$

subject to

$$t \in T = T_1 \times T_2 ... \times T_m, \; x \in X,$$

$$g(t, x) := \frac{1}{2}x^T Qx + x^T(Pt + q) - \varphi(t) \leqslant 0,$$

where

$$\varphi(t) = \min\left\{\frac{1}{2}v^T Qv + v^T(Pt + q) : v \in X, \sum_j b_{kj}v_j \leqslant b_k, \; k = 1, .., \ell\right\}.$$

NUMERICAL EXAMPLES. We illustrate the algorithm with the following examples.

EXAMPLE 1.

$$\min\{f_1(t, x) := x_1^2 + x_2^2 + t^2 - 4t \; : 0 \leqslant t \leqslant 2\},$$

where $x^T = (x_1, x_2)$ solves the convex quadratic program

$$\min_x\left\{f_2(t, x) := x_1^2 + \frac{1}{2}x_2^2 + x_1x_2 + (1 - 3t)x_1 + (1 + t)x_2\right\}$$

subject to

$$2x_1 + x_2 - 2t - 1 \leqslant 0,$$
$$x^T = (x_1, x_2) \geqslant 0.$$

The equivalent one-level problem is

$$\min_{t,x}\{f_1(t,x) := x_1^2 + x_2^2 + t^2 - 4t\}$$

subject to

$$x_1^2 + 0.5x_2^2 + x_1x_2 + (1-3t)x_1 + (t+1)x_2 + 8.5t^2 + t + 0.5 + r(t) \leqslant 0,$$
$$2x_1 + x_2 - 2t - 1 \leqslant 0,\ x^T = (x_1, x_2) \geqslant 0,\ 0 \leqslant t \leqslant 2,$$

where

$$r(t) := \min_{\lambda \in R_+^3} \left\{0.5\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 + 4t\lambda_1 - (5t+1)\lambda_2 + (2-t)\lambda_3\right\}.$$

The approximate function $g_u(t,x)$ has the form

$$\begin{aligned} g_u(t,x) \ &= \ x_1^2 + 0.5x_2^2 + x_1x_2 + (1-3t)x_1 + (t+1)x_2 \\ &\quad +(17u+1)t - 8.5u^2 + 0.5 + r(t). \end{aligned}$$

For computing the lower bound $\beta(S)$ over each simplex S, we have to solve the following two convex quadratic programs, one for each vertex v^S of S:

$$\beta(v^S) := \min_{t,x}\{f_1(t,x) := x_1^2 + x_2^2 + t^2 - 4t\},$$

subject to

$$x_1^2 + 0.5x_2^2 + x_1x_2 + (1-3v^S)x_1 + (v^S+1)x_2$$
$$+(17u^S+1)v - 8.5(u^S)^2 + 0.5 + r(v^S) \leqslant 0,$$
$$2x_1 + x_2 - 2t - 1 \leqslant 0,$$
$$x^T = (x_1, x_2) \geqslant 0,\ t \in S,$$

where u^S can be any point in S. Then we take

$$\beta(S) := \min\{\beta(v^S) :\ v^S \in V(S)\}.$$

Let (t^S, x^S) be an optimal solution correspoding to $\beta(S)$. To compute the upper bound $\alpha(S)$ we solve the convex quadratic program

$$\min_x \left\{x_1^2 + 0.5x_2^2 + x_1x_2 + (1-3t^S)x_1 + (1+t^S)x_2\right\}$$

subject to

$$2x_1 + x_2 - 2t^S - 1 \leqslant 0,$$
$$x^T = (x_1, x_2) \geqslant 0.$$

Let y^S be the optimal solution of this problem Thus (t^S, y^S) is feasible for the bilevel problem Then we take

$$\alpha(S) := f_1(t^S, y^S).$$

Below are details for the first four iterations.

Initialization. The starting interval (one-dimensional simplex) $S_0 := [0, 2]$; approximation point $u^{S_0} = 0$. The lower bound $\beta_0 = \beta(S_0) = -4$ attains at $(t^0, x^0) = (2, 0, 0)$ and $v^0 = 0$. The upper bound $\alpha = 2.25$ attains at $(t^0, y^0) = (2, 2.5, 0)$. Thus the currently best feasible (incumbent) $(\bar{t}^0, \bar{x}^0) = (t^0, y^0) = (2, 2.5, 0)$ and $\Gamma_0 = \{S_0 = [0,\ 2]\}$.

Iteration 0. At this iteration $t^0 = 2$, $v^0 = 0$, $u^0 = 0$. Thus the subdivision point $\omega^0 = 1$ is the midpoint of the interval $[u^0,\ t^0] = [0, 2]$. The simplex $S_0 = [0, 2]$ is bisected into $S_{01} = [0,\ 1]$ and $S_{02} = [1,\ 2]$ via $\omega^0 = 1$. The approximation point $u^0 = \omega^0$. Then $\beta(S_{01}) = -3$. $\beta(S_{02}) = -4$ attain at $(t^{02}, x^{02}) = (2, 0, 0)$, and $v^{02} = 2$.

The currently best upper bound $\alpha_1 = -2$ attains at $(t^1, y^1) = (1, 1, 0)$. The incumbent $(\bar{t}^1, \bar{x}^1) = (1, 1, 0)$ and $\Gamma_1 = \{S_{01}, S_{02}\}$.

Iteration 1. At this iteration $S_1 = S_{02} = [1, 2]$, $t^1 = v^1 = 2$, and $u^1 = 1$. The subdivision point ω^1 thus is 1. The simplex S_1 is bisected into $S_{11} = [1, 1.5]$ and $S_{12} = [1.5,\ 2]$ via $\omega^1 = 1$. The approximation point $u^1 = \omega^1 = 1.5$. Then $\beta(S_{11}) = -3.7500$ attains at $(t^{11}, x^{11}) = (1.5, 0, 0)$, and $v^{11} = 1$. $\beta(S_{12}) = -2.9137$. The currently best upper bound $\alpha_1 = -2$. The incumbent $(\bar{t}^2, \bar{x}^2) = (\bar{t}^1, \bar{x}^1) = (1, 1, 0)$ and $\Gamma_2 = \{S_{01}, S_{11}, S_{12}\}$.

Iteration 2. At this iteration $S_2 = S_{11} = [1, 1.5]$, $t^2 = 1.5$, $v^2 = 1$, and $u^2 = 1.5$. The subdivision point ω^2 thus is 1.25. The the simplex S_2 is bisected into $S_{21} = [1, 1.25]$ and $S_{22} = [1.25,\ 1.5]$ via $\omega^2 = 1.25$. The approximation point $u^2 = \omega^2$. Then $\beta(S_{21}) = -3.3640$ attains at $t^{21} = (1.25, x^{21}) = (0.2711, 0)$ and $v^{21} = 1$. $\beta(S_{22}) = -2.2037$.

The currently upper bound $\alpha_3 = -2$. The incumbent $(\bar{t}^3, \bar{x}^3) = (\bar{t}^1, \bar{x}^1) = (1, 1, 0)$ and

$$\Gamma_3 = \{S_{01}, S_{12}, S_{21}, S_{22}\}.$$

The algorithm terminates after 16- iterations yielding an $\epsilon = 0.06$-optimal solution $(t, x_1, x_2) = (0.8438, 0.7657, 0)$ with the ϵ-optimal value $f_1(t, x_1, x_2) = -2.0769$.

In this example, subdivision simplices are intervals and all subdivison points are the midpoints of the subdivision intervals. So in this case the adaptive subdivision coincides with the exhaustive bisection. In the next example the adaptive simplicial subdivision is not necessarily exhaustive.

EXAMPLE 2.

$$\min\{f_1(t,x) := x_1^2 + x_3^2 - x_1x_3 - 4x_2 - 7t_1 + 4t_2 : t^T = (t_1, t_2) \geqslant 0,\ t_1 + t_2 \leqslant 1\},$$

where $x^T := (x_1, x_2, x_3)$ solves the quadratic program

$$\min_x\{f_2(t,x) := x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_2 + (1 - 3t_1)x_1 + (1 + t_2)x_2\},$$

subject to

$$\begin{aligned}
&2x_1 + x_2 - x_3 + t_1 - 2t_2 + 2 \leqslant 0,\\
&x^T = (x_1, x_2, x_3) \geqslant 0.
\end{aligned}$$

The equivalent one-level problem then takes

$$\min_{t,x}\{f_1(t,x) := x_1^2 + x_3^2 - x_1x_3 - 4x_2 - 7t_1 + 4t_2\}$$

subject to

$$\begin{aligned}
&x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_2 + (1 - 3t_1)x_1 + (1 + t_2)x_2\\
&+\frac{9}{2}t_1^2 + t_2^2 + 3t_1t_2 + t_2 + \frac{1}{2} + r(t) \leqslant 0,\\
&2x_1 + x_2 - x_3 + t_1 - 2t_2 + 2 \leqslant 0,\ x^T = (x_1, x_2, x_3) \geqslant 0,\\
&t^T = (t_1, t_2) \geqslant 0,\ t_1 + t_2 \leqslant 1,
\end{aligned}$$

where

$$\begin{aligned}
r(t) \ := \ \min_{\lambda \in R_+^4} \Big\{ &\frac{3}{2}\lambda_1^2 + \frac{1}{2}\lambda_2^2 + \lambda_3^2 + \frac{1}{2}\lambda_4^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 + \lambda_1\lambda_4\\
&- (4t_1 - 2t_2 + 1)\lambda_1 + (3t_1 + t_2)\lambda_2 - (3t_1 + 2t_2 + 1)\lambda_3 \Big\}.
\end{aligned}$$

The approximate function $g_u(t, x)$ takes the form

$$\begin{aligned}
g_u(t,x) \ = \ &x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_2 + (1 - 3t_1)x_1 + (1 + t_2)x_2\\
&+(9u_1+3u_2)t_1+(3u_1+2u_2+1)t_2-\frac{9}{2}u_1^2-u_2^2-3u_1u_2+\frac{1}{2}+r(t).
\end{aligned}$$

Table 1.

Iter	Subdivision simplices		
1	(0.000, 0.000);	(0.000, 1.000);	(1.000, 0.000)
2	(0.000, 0.000);	(0.000, 1.000);	(0.500, 0.500)
3	(0.000, 0.000);	(0.500, 0.500);	(1.000, 0.000)
4	(0.000, 0.000);	(0.500, 0.500);	(0.750, 0.250)
5	(0.250, 0.250);	(0.500, 0.500);	(0.750, 0.250)
6	(0.250, 0.250);	(0.500, 0.500);	(0.625, 0.375)
7	(0.437, 0.312);	(0.500, 0.500);	(0.625, 0.375)
8	(0.531, 0.343);	(0.500, 0.500);	(0.625, 0.375)
9	(0.250, 0.250);	(0.500, 0.500);	(0.437, 0.312)
10	(0.531, 0.343);	(0.500, 0.500);	(0.562, 0.437)
11	(0.437, 0.312);	(0.500, 0.500);	(0.531, 0.343)
12	(0.437, 0.312);	(0.500, 0.500);	(0.515, 0.421)
13	(0.531, 0.343);	(0.562, 0.437);	(0.625, 0.375)
14	(0.375, 0.375);	(0.500, 0.500);	(0.437, 0.312)
15	(0.531, 0.343);	(0.562, 0.437);	(0.593, 0.406)
16	(0.250, 0.250);	(0.000, 1.000);	(0.500, 0.500)
17	(0.531, 0.343);	(0.515, 0.421);	(0.562, 0.437)
18	(0.531, 0.343);	(0.593, 0.406);	(0.625, 0.375)
19	(0.531, 0.343);	(0.593, 0.406);	(0.609, 0.390)

For computing the lower bound $\beta(S)$ over each simplex S, we have to solve the following three convex quadratic programs, one for each vertex v^S of S:

$$\beta(v^S) := \min_{t,x}\{f_1(t,x) : x_1^2 + x_3^2 - x_1x_3 - 4x_2 - 7t_1 + 4t_2\}$$

subject to

$$\begin{aligned}
&x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_2 + (1-3v_1^S)x_1 + (1+v_2^S)x_2 \\
&+(9u_1^S + 3u_2^S)v_1^S + (3u_1^S + 2u_2^S + 1)v_2^S \\
&-\frac{9}{2}(u_1^S)^2 - (u_2^S)^2 - 3u_1^Su_2^S + \frac{1}{2} + r(v^S) \leqslant 0, \\
&2x_1 + x_2 - x_3 + t_1 - 2t_2 + 2 \leqslant 0, \\
&x^T = (x_1, x_2, x_3) \geqslant 0,\ t \in S,
\end{aligned}$$

where u^S can be an arbitrary point in S. Let (t^{v^S}, x^{v^S}) be an optimal solution of this problem. Let (t^S, x^S) be an optimal solution correspoding to $\beta(S)$. To compute the upper bound $\alpha(S)$ we solve the following convex quadratic program where t^S is

Table 2.

Iter	α	β	γ	u^S	ω^S
1	4.000	0.000	1	(0.000, 0.000)	(0.500, 0.500)
2	0.750	0.149	2	(0.500, 0.500)	(0.250, 0.250)
3	0.750	0.500	2	(0.500, 0.500)	(0.750, 0.250)
4	0.750	0.500	2	(0.000, 0.000)	(0.250, 0.250)
5	0.750	0.500	2	(0.500, 0.500)	(0.625, 0.375)
6	0.640	0.504	3	(0.250, 0.250)	(0.437, 0.312)
7	0.640	0.555	4	(0.437, 0.312)	(0.531, 0.343)
8	0.640	0.578	5	(0.531, 0.343)	(0.562, 0.437)
9	0.640	0.587	6	(0.437, 0.312)	(0.375, 0.375)
10	0.640	0.592	6	(0.562, 0.437)	(0.515, 0.421)
11	0.640	0.596	6	(0.531, 0.343)	(0.515, 0.421)
12	0.640	0.596	6	(0.515, 0.421)	(0.468, 0.406)
13	0.640	0.604	5	(0.562, 0.437)	(0.593, 0.406)
14	0.640	0.604	6	(0.375, 0.375)	(0.468, 0.406)
15	0.640	0.608	5	(0.593, 0.406)	(0.546, 0.390)
16	0.640	0.610	6	(0.250, 0.250)	(0.375, 0.375)
17	0.640	0.612	5	(0.515, 0.421)	(0.546, 0.390)
18	0.640	0.620	5	(0.593, 0.406)	(0.609, 0.390)
19	0.638	0.622	1	(0.609, 0.390)	(0.570, 0.367)
20	0.638	0.625	0		

fixed.

$$\min_x \left\{ x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_2 + (1 - 3t_1^S)x_1 + (1 + t_2^S)x_2 \right\}$$

subject to

$$2x_1 + x_2 - x_3 + t_1^S - 2t_2^S + 2 \leqslant 0,$$
$$x^T = (x_1, x_2, x_3) \geqslant 0.$$

Denote by y^S the unique solution of this problem. Thus (t^S, y^S) is a feasible for the bilevel problem To get the upper bound for the simplex S we take

$$\alpha(S) := f_1(t^S, y^S).$$

The computed results are summarized in the two tables below. The algorithm terminated at iteration number 19 yielding an ϵ -global optimal solution

$$x^T = (0.000, 0.000, 1.828), \quad t^T = (0.609, 0.391)$$

with $\epsilon = 0.01$.

In Tables 1 and 2 we use the following headings:

- $iter$: iteration number
- α, β: currently best upper and lower bounds
- γ: number the simplices to be restored at each iteration
- u^S and ω^S: approximation and subdivision points for simplex S.

5. Conclusions

We have approximated a merit function of a parameterized convex quadratic problem by saddle functions. Using this approximation we have proposed an algorithm for finding a global optimal solution to the linearly constrained bilevel convex quadratic problem. The global search of the proposed algorithm is performed in the space of parameters via an adaptive simplicial subdivision. The algorithm thus is designed for problems where the number of the parameters is relatively small. The number of the variables of the second-level problem may be larger.

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